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The Effect of Magnetic Field on Thermal Shock Problem for a Fiber-Reinforced Anisotropic Half-Space Using Green-Naghdi's Theory

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This article presents a two-dimensional problem of generalized thermoelasticity for a fiber-reinforcement anisotropic half-space under a thermal shock at its upper surface. The effects of initial stress and rotation are both studied. Green and Naghdi's theory of thermoelasticity is employed to study the present problem. The inclusion of reinforcement anisotropic elastic parameter is made and two additional terms are added to the displacement equation. The problem is solved numerically by using the finite element method. Numerical results for displacements, stresses and temperature are given and presented graphically in different positions. Comparisons are made for different values of the magnetic field. The inclusion of the reinforcement parameters is also investigated.

Keywords: Magneto-Thermoelasticity, Green-Naghdi's Theory (GNII), Fiber-Reinforced, Half-Space, Thermal Shock, Finite Element Method.

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1. INTRODUCTION

Due to their low weight and high strength, the fibre-reinforced composite materials are widely used in a variety of structures. The analysis of stress and deformation of fiber-reinforced composites has been an important subject of solid mechanics for last three decades. Some investigators such as Spencer,[1] Sun et al.,[2] Aboudi,[3] Bunsell,[4] Mallick,[5] Bunsell and Renard,[6] Liu[7] and others, did pioneer works on the subject. In the last three decades, the analysis of stress and deformation of fiber-reinforced composite materials has been an important research area of solid mechanics. Also, one can find some work on generalized thermoelsticity theories in the literature.

The first relaxation time is firstly used by Lord and Shulman[8] in their theory of generalized thermoelasticity for an isotropic body. An extension of this theory for an anisotropic body is introduced by Dhaliwal and Sherief.[9] In this theory, a modified law of heat conduction including both the heat flux and its time derivatives replaces the conventional Fourier's law. The heat equation associated with this theory is hyperbolic and hence eliminates the paradox of infinite speeds of propagation inherent in both coupled and uncoupled theories of thermoelasticity.

The characteristics of material response for thermal phenomena are presented by Green and Naghdi[10] based on three types of constitutive response functions labeled as I, II, III. The constitutive equations in these three types are linearized. For example, Type I is the same as the classical heat conduction equation (based on Fourier's law), whereas the linearized version of Type II theory permits propagation of thermal waves at finite speed and involves no dissipation of thermal energy. Further, Type III is considered as the most general, which (formally) includes the others as particular instances. It involves a thermal damping term and predicts an infinite speed of thermal propagation. Another generalization of the heat-flux constitutive law was proposed by Coleman and Gurtin.[11] It is also referred to as a theory of heat conduction with thermal memory because of the presence of a time-convolution integral. In fact, the Coleman-Gurtin relation can be considered as the most general model among all the others cited so far and the linear model of Gurtin-Pipkin[12] is considered as one of its special cases. Though several problems relating to generalized thermoelasticity theory of Types II and III have been investigated.[13–21] Recently, variants problems in waves are studied.[22–24] Other forms are described for example in the Refs. [25–27].

The theory of magneto-thermoelasticity has drawn the attention of many researchers in recent years. It deals

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with the interactions between strain, temperature and electromagnetic fields. It possesses many extensive applications in diverse fields, such as geophysics, for understanding of the effect of the Earth's magnetic field on seismic waves. In addition, the damping of acoustic waves in a magnetic field, the emission of electromagnetic radiation from nuclear devices, the development of a highly sensitive superconducting magnetometers, electrical power engineering, optics are considered as extensive applications of this theory.

The object of this article is to present a two-dimensional thermal-shock problem of a fiber-reinforcement anisotropic half-space. The thermoelastic interactions in the half-space is studied due to a uniform magnetic field in the context of the generalized thermoelasticity II proposed by Green and Naghdi.[10] The finite element solution for the coupled governing equations is obtained. Numerical results are provided to show the influence of the magnetic field on the temperature, displacements and stresses.

2. FORMULATION OF THE PROBLEM

Let us consider the problem of a thermoelastic half-space ($x \geq 0$). A magnetic field with constant intensity $\mathbf{H} = (0, 0, H_0)$ is acting parallel to the boundary plane (taken as the direction of the z-axis). The surface of the half-space is subjected to a thermal shock which is a function of y and t. Thus, all the quantities considered will be functions of the time variable t, and of the coordinates x and y. Let us begin this consideration with the linearized equations of electro-dynamics of slowly moving medium[28]

$$\mathbf{J} = \operatorname{curl} \mathbf{h} - \varepsilon_0 \dot{\mathbf{E}} \tag{1}$$

$$\operatorname{curl} \mathbf{E} = -\mu_0 \dot{\mathbf{h}} \tag{2}$$

$$\mathbf{E} = -\mu_0 (\dot{\mathbf{u}} \times \mathbf{H}) \tag{3}$$

$$\nabla \cdot \mathbf{h} = 0 \tag{4}$$

where μ_0 is the magnetic permeability; ε_0 is the electric permeability, $\dot{\mathbf{u}}$ is the particle velocity of the medium, $\mathbf{h}$ is the induced magnetic field vector, $\mathbf{E}$ is the induced electric field vector and $\mathbf{J}$ is the current density vector. The comma notation is used for spatial derivatives and superimposed dot represents time differentiation.

These equations are supplemented by the displacement equations of the theory of elasticity, taking into consideration the Lorentz force F_i to give

$$\sigma_{ij,j} + F_i = \rho \ddot{u}_i \tag{5}$$

$$F_i = \mu_0 (\mathbf{J} \times \mathbf{H})_i \tag{6}$$

where σ_{ij} is the stress tensor, u_i are the displacement components and ρ is the mass density. The constitutive equation for a fiber-reinforced linearly thermoelastic anisotropic medium whose preferred direction is that of a unit vector $\mathbf{a}$ is

$$\begin{aligned}\sigma_{ij} &= \lambda e_{kk}\delta_{ij} + 2\mu_T e_{ij} + \alpha(a_k a_m e_{km}\delta_{ij} + a_i a_j e_{kk}) \\ &\quad + 2(\mu_L - \mu_T)(a_i a_k e_{kj} + a_j a_k e_{ki}) + \beta a_k a_m e_{km} a_i a_j \\ &\quad - \beta_{ij}(T - T_0)\delta_{ij}, \quad i,\ j,\ k,\ m = 1, 2, 3\end{aligned} \tag{7}$$

where T is the temperature change of a material particle, T_0 is the reference uniform temperature of the body, β_{ij} is the thermal elastic coupling tensor, δ_{ij} is the Kronecker delta, λ and μ_T are elastic parameters, α, β and $(\mu_L - \mu_T)$ are reinforced anisotropic elastic parameters and $\mathbf{a} \equiv (a_1, a_2, a_3)$, $a_1^2 + a_2^2 + a_3^2 = 1$. The heat conduction equation

$$K^* T_{ij} + K_{ij}\dot{T}_{ij} = \rho c_e \ddot{T} + T_0 \ddot{u}_{i,j} \tag{8}$$

where c_e is the specific heat at constant strain, K_{ij} are the thermal conductivity components and K^* is the material characteristic of the theory. For the problem of a thermoelastic half-space ($x \geq 0$) in the context of Green-Naghdi's (GNII) generalized thermoelasticity theory (without energy dissipation), all the considered functions will be depend on the time t and the coordinates x and y. Thus, the displacement components u_i will be

$$u_x = u(x, y, t), \quad u_y = v(x, y, t), \quad u_z = 0 \tag{9}$$

Let us choose the fibre-direction as $\mathbf{a} \equiv (1, 0, 0)$ so that the preferred direction is the x-axis and Eqs. (5)–(7) are simplified as

$$\begin{aligned}\sigma_{xx} &= (\lambda + 2\alpha + 4\mu_L - 2\mu_T + \beta)\frac{\partial u}{\partial x} + (\lambda + \alpha)\frac{\partial v}{\partial y} \\ &\quad - \beta_{11}(T - T_0)\end{aligned} \tag{10}$$

$$\sigma_{yy} = (\lambda + \alpha)\frac{\partial u}{\partial x} + (\lambda + 2\mu_T)\frac{\partial v}{\partial y} - \beta_{22}(T - T_0) \tag{11}$$

$$\sigma_{xy} = \mu_L\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \tag{12}$$

$$F_x = \mu_0 H_0^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} - \varepsilon_0\mu_0\frac{\partial^2 u}{\partial t^2}\right) \tag{13}$$

$$F_y = \mu_0 H_0^2\left(\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial y^2} - \varepsilon_0\mu_0\frac{\partial^2 v}{\partial t^2}\right) \tag{14}$$

$$\begin{aligned}&(A_{11} + \rho R_H^2)\frac{\partial^2 u}{\partial x^2} + (A_{12} + \rho R_H^2)\frac{\partial^2 v}{\partial x \partial y} + A_{13}\frac{\partial^2 u}{\partial y^2} - \beta_{11}\frac{\partial T}{\partial x} \\ &\quad = \rho\left(1 + \frac{R_H^2}{c^2}\right)\frac{\partial^2 u}{\partial t^2}\end{aligned} \tag{15}$$

$$\begin{aligned}&(A_{22} + \rho R_H^2)\frac{\partial^2 v}{\partial y^2} + (A_{12} + \rho R_H^2)\frac{\partial^2 u}{\partial x \partial y} + A_{13}\frac{\partial^2 v}{\partial x^2} - \beta_{22}\frac{\partial T}{\partial y} \\ &\quad = \rho\left(1 + \frac{R_H^2}{c^2}\right)\frac{\partial^2 v}{\partial t^2}\end{aligned} \tag{16}$$

$$K^*\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) = \rho c_e \frac{\partial^2 T}{\partial t^2} + T_0 \frac{\partial^2}{\partial t^2}\left(\beta_{11}\frac{\partial u}{\partial x} + \beta_{22}\frac{\partial v}{\partial y}\right) \tag{17}$$

where

$$A_{11}=\lambda+2(\alpha+\mu_T)+4(\mu_L-\mu_T)+\beta, \quad R_H^2=\frac{\mu_0 H_0^2}{\rho}$$
$$A_{12}=\alpha+\lambda+\mu_L, \quad A_{13}=\mu_L, \quad A_{22}=\lambda+2\mu_T$$
$$c^2=\frac{1}{\varepsilon_0\mu_0}, \quad \beta_{11}=(2\lambda+3\alpha+4\mu_L-2\mu_T+\beta)\alpha_{11} \tag{18}$$
$$\beta_{22}=(2\lambda+\alpha)\alpha_{11}+(\lambda+2\mu_T)\alpha_{22}$$

in which α_{11}, α_{22} are coefficients of linear thermal expansion. For convenience, the following non-dimensional variables are used:

$$(x',y',u',v')=c_1\chi(x,y,u,v), \quad t'=c_1^2\chi t$$
$$T'=\frac{\beta_{11}(T-T_0)}{\rho c_1^2}, \quad \chi=\frac{\rho c_e}{K_{11}} \tag{19}$$
$$(\sigma'_{xx},\sigma'_{xy},\sigma'_{yy})=\frac{1}{\rho c_1^2}(\sigma_{xx},\sigma_{xy},\sigma_{yy}), \quad h'=\frac{h}{H_0}, \quad c_1^2=\frac{A_{11}}{H_0}$$

In terms of the non-dimensional quantities defined in Eq. (19), the above governing equations reduce to (dropping the dashed for convenience)

$$\sigma_{xx}=\frac{\partial u}{\partial x}+B_1\frac{\partial v}{\partial y}-T \tag{20}$$

$$\sigma_{yy}=B_1\frac{\partial u}{\partial x}+B_2\frac{\partial v}{\partial y}-B_3T \tag{21}$$

$$\sigma_{xy}=B_4\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right) \tag{22}$$

$$\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{xy}}{\partial y}+\eta\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x\partial y}\right)=\xi\frac{\partial^2 u}{\partial t^2} \tag{23}$$

$$\frac{\partial\sigma_{xy}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+\eta\left(\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 v}{\partial y^2}\right)=\xi\frac{\partial^2 v}{\partial t^2} \tag{24}$$

$$c_T^2\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)=\frac{\partial^2 T}{\partial t^2}+\frac{\partial^2}{\partial t^2}\left(\varepsilon_0\frac{\partial u}{\partial x}+\varepsilon_1\frac{\partial v}{\partial y}\right) \tag{25}$$

where

$$(B_1,B_2,B_4)=\frac{1}{A_{11}}(A_{12},A_{22},A_{13}), \quad B_3=\frac{\beta_{22}}{\beta_{11}}$$
$$\xi=1+\frac{R_H^2}{c^2}, \quad c_T^2=\frac{K^*}{\rho c_e}, \quad \eta=\frac{R_H^2}{c_1^2} \tag{26}$$
$$(\varepsilon_0,\varepsilon_1)=\frac{T_0\beta_{11}c^2}{A_{11}\rho c_e}(\beta_{11},\beta_{22})$$

3. FINITE ELEMENT FORMULATION

The Finite element method is a powerful technique originally developed for numerical solution of complex problems in structural mechanics, and it remains the method of choice for complex systems. A further benefit of this method is that it allows physical effects to be visualized and quantified regardless of experimental limitations. On the other hand, the finite element method in different generalized thermoelastic problems has been applied by many authors (see for instant Refs. [28]–[33]).

In this section, the governing equations of generalized thermoelasticity based upon Green and Naghdi's theory are summarized, using the corresponding finite element equations. In the finite element method, the displacement components u, v and the temperature T are related to the corresponding nodal values by

$$u=\sum_{i=1}^{m}N_i u_i(t), \quad v=\sum_{i=1}^{m}N_i v_i(t), \quad T=\sum_{i=1}^{m}N_i T_i(t) \tag{27}$$

where m denotes the number of nodes per element, and N_i are the shape functions. The eight-node isoparametric, quadrilateral element is used for displacement components and temperature calculations. The weighting functions and the shape functions coincide. Thus,

$$\delta u=\sum_{i=1}^{m}N_i\delta u_i, \quad \delta v=\sum_{i=1}^{m}N_i\delta v_i, \quad \delta T=\sum_{i=1}^{m}N_i\delta T_i \tag{28}$$

It should be noted that appropriate boundary conditions associated with the governing equations, Eqs. (23)–(25) must be adopted in order to properly formulate a problem. Boundary conditions are either essential (geometric) or natural (traction) types. Essential conditions are prescribed displacements u, v and temperature T while, the natural boundary conditions are prescribed tractions and heat flux. They expressed as

$$\sigma_{xx}n_x+\sigma_{xy}n_y=\bar{\tau}_x, \quad \sigma_{xy}n_x+\sigma_{yy}n_y=\bar{\tau}_y$$
$$q_xn_x+q_yn_y=\bar{q} \tag{29}$$

where n_x and n_y are direction cosines of the outward unit normal vector at the boundary, $\bar{\tau}_x$ and $\bar{\tau}_y$ are the given tractions values, and $\bar{q}$ is the given surface flux.

In the absence of body force, the governing equations are multiplied by weighting functions and then are integrated over the spatial domain ϖ with the boundary Γ. Applying integration by parts and making use of the divergence theorem reduce the order of the spatial derivatives

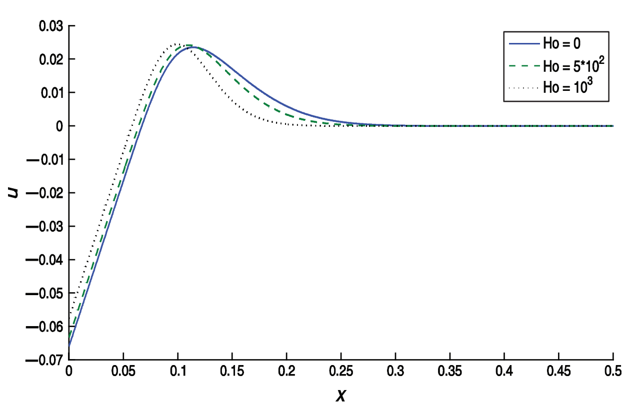

Fig. 1. Variation of horizontal displacement u with distance x.

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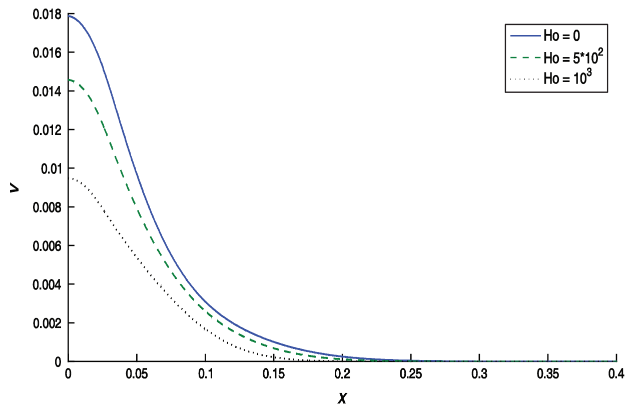

Fig. 2. Variation of vertical displacement v with distance x.

and allows for the application of the boundary conditions. Thus, the finite element equations corresponding to Eqs. (23)–(25) can be obtained as

$$\int_{\Gamma}\begin{Bmatrix}\delta u\bar{\tau}_x\\ \delta v\bar{\tau}_y\\ \delta T\bar{q}\end{Bmatrix}d\Gamma = \int_{\varpi}\begin{Bmatrix}\dfrac{\partial\delta u}{\partial x}\sigma_{xx}+\dfrac{\partial\delta u}{\partial y}\sigma_{xy}\\ \dfrac{\partial\delta v}{\partial x}\sigma_{xy}+\dfrac{\partial\delta v}{\partial y}\sigma_{yy}\\ c_T^2\left(\dfrac{\partial\delta T}{\partial x}\dfrac{\partial T}{\partial x}+\dfrac{\partial\delta T}{\partial y}\dfrac{\partial T}{\partial y}\right)\end{Bmatrix}d\varpi + \int_{\varpi}\begin{Bmatrix}\delta u\left[\dfrac{\partial^2 u}{\partial t^2}-\eta\left(\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}\right)\right]\\ \delta v\left[\dfrac{\partial^2 v}{\partial t^2}-\eta\left(\dfrac{\partial^2 v}{\partial x^2}+\dfrac{\partial^2 v}{\partial y^2}\right)\right]\\ \delta T\left[\dfrac{\partial^2 T}{\partial t^2}+\dfrac{\partial^2}{\partial t^2}\left(\varepsilon_0\dfrac{\partial u}{\partial x}+\varepsilon_1\dfrac{\partial v}{\partial y}\right)\right]\end{Bmatrix}d\varpi \tag{30}$$

Symbolically, the discredited equations of Eq. (23) can be written as

$$M\ddot{d}+c\dot{d}+Kd = F^{\text{ext}} \tag{31}$$

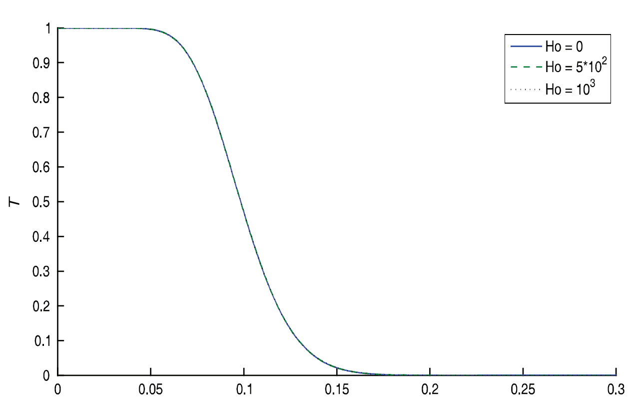

Fig. 3. Variation of temperature T with distance x.

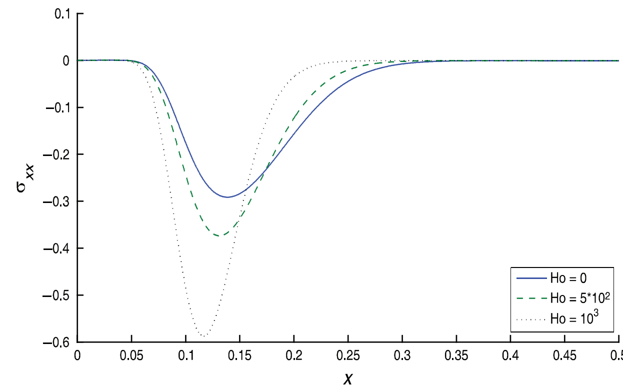

Fig. 4. Variation of longitudinal stress component σ_{xx} with distance x.

where M, C, K, and F^{ext} represent the mass, damping, stiffness matrices, and external force vectors, respectively, $d=[u\quad v\quad T]^T$. On the other hand, the time derivatives of the unknown variables have to be determined by the Newmark time integration method (see Wriggers[34]).

4. NUMERICAL EXAMPLE

To study the effect of magnetic field on wave propagation, the following physical constants for generalized fibre-reinforced thermoelastic material are used.

$$\rho=2660\ \text{kg/m}^3,\quad \lambda=5.65\times10^{10}\ \text{N/m}^2$$
$$\mu_T=2.46\times10^{10}\ \text{N/m}^2,\quad \mu_L=5.66\times10^{10}\ \text{N/m}^2$$
$$\alpha=-1.28\times10^{10}\ \text{N/m}^2,\quad \beta=220.90\times10^{10}\ \text{N/m}^2$$
$$\alpha_{11}=0.017\times10^{-4}\ \text{deg}^{-1},\quad l=1,\quad \alpha_{22}=0.015\times10^{-4}\ \text{deg}^{-1}$$
$$c_e=0.787\times10^3\ \text{Jkg}^{-1}\text{deg}^{-1},\quad T_0=293\ \text{k},\quad T_1=1$$
$$K_{11}=0.0921\times10^3\ \text{Jm}^{-1}\text{s}^{-1}\text{deg}^{-1}$$
$$K_{22}=0.0963\times10^3\ \text{Jm}^{-1}\text{s}^{-1}\text{deg}^{-1}$$

The numerical applications will be carried out for the displacements u and v, temperature T, and stresses σ_{xx}, σ_{xy} and σ_{yy} that being reported herein at $y=0.5$, $t=0.2$

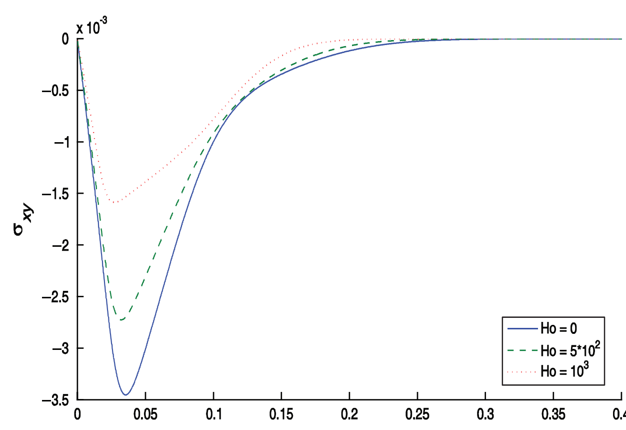

Fig. 5. Variation of tangential stress component σ_{xy} with distance x.

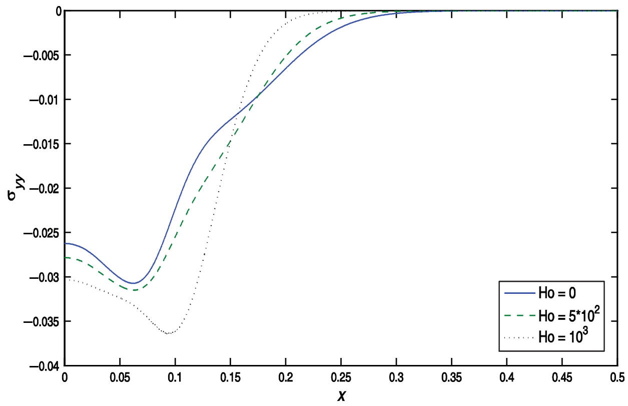

Fig. 6. Variation of normal stress component σ_{yy} with distance x.

and for different values of primary magnetic stress H_0. Effect of the primary magnetic field $H_0 = 0$, $H_0 = 5 \times 10^2$ and $H_0 = 10^3$ on the field quantities are showed in Figures 1–6. The distributions of u, v, T, σ_{xx}, σ_{xy} and σ_{yy} are presented through the longitudinal x-direction.

All quantities, except the temperature, are very sensitive to the variation of the primary magnetic field H_0. The magnetic field itself has no effect on other quantities for $x > 0.3$. In the interval $0 \leq x < 0.3$, as H_0 increases the vertical displacement v decreases while the tangential stress σ_{xy} decreases. The temperature T decreases as x increases and it may be independent of H_0. However, the values of the longitudinal and normal stresses σ_{xx} and σ_{xy} as well as the horizontal displacement u oscillate randomly with distance x.

Figure 1 shows that the horizontal displacement u is no longer increasing and it is decreasing monotonically with maximum values at $x \cong 0.1$. Figure 2 shows that the vertical displacement v decreases directly as x increases. Both u and v reach to zero when $x \cong 0.3$. Once again, the temperature T is independent of H_0 and it decreases as x increases and vanishes when $x \cong 0.3$ as shown in Figure 3. Figure 4 shows that the longitudinal stress σ_{xx} is no longer decreasing and it is increasing monotonically with minimum values at $0.11 < x < 0.13$. The same behaviors are illustrated by σ_{xy} and σ_{yy} in Figures 5 and 6, respectively. The two stresses have minimum values at $0.02 < x < 0.04$ and $0.05 < x < 0.1$, respectively.

5. CONCLUSION

In this paper, a finite element scheme has been described to solve the present shock-problem. The results obtained show that behavior of the displacements, temperature, and stresses may change significantly by reason of influence of the primary magnetic field. As observed of the plots of all quantities, the primary magnetic field has a significant effect on all the studied quantities, except perhaps the temperature. The thermoelastic horizontal displacement first increases then decreases, but the stresses first decrease then increases. However, the vertical displacement and temperature decrease directly to reach zero values along the horizontal direction.

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